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Bending of Beams With Elastically Non-Linear Coatings

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Young's moduli of thermally sprayed materials are known to be substantially lower than those of well-sintered materials, due to elastic openings and partial closings of microcracks. Under compressive stresses some microcrack faces come into contact, which leads to the increase of Young's moduli and to non-linear elastic behavior. The bending of beams with elastically non-linear coatings on the compression face caused by external moments or residual stresses is analyzed theoretically. Examples of stress distributions are given for a model ceramic coating. Different coating and substrate elastic properties and thicknesses are considered. The difference between the presented non-linear and previously used linear solutions is shown to be important at high compressive stresses, especially in ceramic coatings with high densities of microcracks.

Keywords ceramic coatings, closing of microcracks, compressive stresses, external and residual stresses, non-linear stress-strain relation, pure bending theory

1. Introduction

The coatings prepared by thermal spraying have between 2-20 times smaller Young's moduli than the corresponding well-sintered materials. Moreover, they have different Young's moduli in the directions parallel and perpendicular to the surface. These effects are more pronounced in ceramic than in metal coatings.[1]

The effects were explained in a series of theoretical papers to be mainly due to small elastic openings and partial closings of the intersplat and intrasplat microcracks present in the coatings.[2-5] The microcracks are assumed to be sufficiently opened (see Ref. 2-5) so that the surfaces do not come into contact under compressive stresses, and the same values of the elastic constants under tension and compression follow.

A possible effect of compressive stresses on Young's moduli of thermally sprayed materials was discussed in Ref. 6. The surfaces of a crack may come into contact under compressive stress, depending on the crack aspect ratio. The increase of Young's moduli with increasing compressive stresses, i.e., non-linear elastic behavior in compression of thermally sprayed materials, was predicted in Ref. 6. A similar effect is well known in geophysics as the effect of hydrostatic pressure on the velocity of ultrasonic waves and on the elastic constants of rocks containing cracks.[7-9] The velocity of ultrasonic waves in different directions in a sphere made of plasma-sprayed alumina under hydrostatic pressure between 0 and 400 MPa has recently been measured in the Institute of Geophysics in Prague by T. Lokajicek and co-workers; the corresponding tangent Young's moduli increased about three times (unpublished research).

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The elastic behavior of sprayed materials under compressive stresses was recently modeled in Ref. 10 within the framework of the non-linear anisotropic theory of elasticity. Physical non-linearity with small deformations was assumed. Stress-strain relations were based on a simplified modeling of the closing of microcracks under compressive stresses in continuum.

Compressive stresses appear, e.g., during bending of beams with sprayed coatings on the compression face. Moreover, high residual compressive stresses appear in some coatings. The bending of beams with coatings, either due to external forces or residual stresses, was studied analytically within linear elasticity theory in a number of papers, e.g., in Ref. 11-13. Such theoretical results were used especially in the evaluation of bending experiments for determining elastic constants or residual stresses in coatings. A consideration of the effect of compressive stresses on elastic constants may improve these results, especially in cases with high compressive stresses.

For this paper, the bending of a beam with elastically non-

Nomenclature	
A, B, A_0, B_0, C, n	constants
e	elastic strain
E	Young's modulus, Pa
h	thickness, m
$k = h_S/h$, $K = E_S/E_1$, $m = M/(h^2E_1)$	dimensionless parameters
M	moment (per unit thickness), N
x, y, z	Cartesian coordinates, m
ε	initial strain
σ	stress, Pa
Subscripts	
L	limit
S	substrate
SEC	secant
T	total
TAN	tangent
1	for small stresses

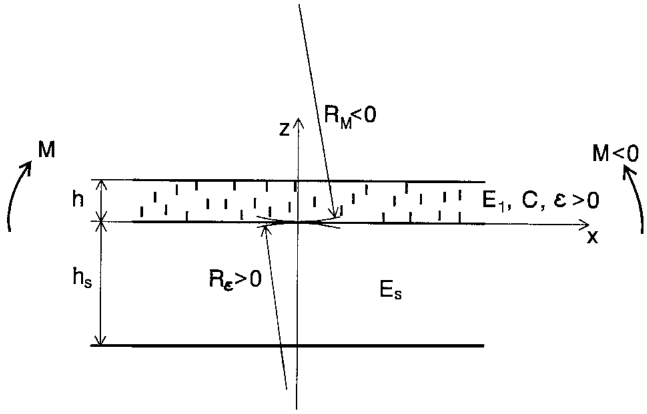

Fig. 1 Substrate with coating on compression side

linear coating, with stress-strain relations based on Ref. 10, was studied. The problem is formulated analytically, but the resulting system of equilibrium equations is solved numerically. To obtain the most general results, dimensionless quantities are used throughout.

2. Bending Due to External Moment and Residual Stresses

2.1 Governing Equations

Pure elastic bending due to external moment M (per unit thickness in the y direction) (Fig.1) and to homogeneous initial deformation ε in the coating will be studied as a plane stress problem. The initial deformation may be connected with the secondary (differential thermal contraction) residual stresses caused by cooling of the coating with substrate from temperature T to a reference temperature T_0, $T > T_0$. Homogeneous initial deformations in the x direction in the coating and the substrate are $\varepsilon = \alpha\ (T_0 - T)$, $\varepsilon_S = \alpha_S\ (T_0 - T)$, where α and α_S are the constant thermal expansion coefficients of the coating and substrate, respectively. For simplicity, we assume in the further text that in the coating $\varepsilon = (\alpha_S - \alpha)(T - T_0)$ while in the substrate $\varepsilon_S = 0$, without loss of generality.

In view of the compatibility equations, the total strain in the x direction, $e^T_{xx} = e_T$, must be a linear function of coordinate z and will be written as

$$e_T = \mathrm{A}\ (z/h) + \mathrm{B}, \quad -h_S \le z \le h, \qquad \text{(Eq 1)}$$

where h and h_S are the thicknesses of the coating and substrate, respectively. A and B are dimensionless constants, which can be determined from the equilibrium conditions.

Elastic deformations in the x direction in the coating e and in the substrate e_S are given by

$$e = e_T - \varepsilon = \mathrm{A}\ (z/h) + \mathrm{B} - \varepsilon, \quad 0 < z \le h, \qquad \text{(Eq 2)}$$

$$e_S = e_T = \mathrm{A}\ (z/h) + \mathrm{B}, \quad \mathrm{h_S} \le z < 0$$

Next, the elastic stress-strain relations must be specified. A homogeneous isotropic elastically linear substrate with Young's modulus E_S is considered by assuming

$$\sigma_S = E_S\ e_S, \quad -h_S \le z < 0 \qquad \text{(Eq 3)}$$

A non-linear elastic stress-strain relation for the coating under compressive stresses, $\sigma < 0$, and elastic contractions, $e < 0$, is taken from Ref. 10 in the form

$$e = (\sigma/E_1) + \mathrm{C}\ (\sigma/E_1)^2, \quad 0 < z \le h, \qquad \text{(Eq 4)}$$

where E_1 is Young's modulus of the coating for $\sigma \to 0$ (for small stresses) and C is a dimensionless positive material constant characterizing the non-linearity of the coating.

The corresponding stress-strain relation $\sigma(e)$ can be obtained from Eq 4 as a solution of the quadratic algebraic equation

$$(\sigma/E_1) = [1/(2\mathrm{C})]\ [-1 + (1 + 4\mathrm{C}e)^{1/2}] \qquad \text{(Eq 5)}$$

The second solution of the quadratic equation, with the minus sign in front of the bracket $(1 + 4\mathrm{C}e)^{1/2}$, has no physical meaning.

The secant modulus of the coating, $E_{SEC} = \sigma/e$, depends on stress σ as

$$E_{SEC} = E_1/[1 + \mathrm{C}\ (\sigma/E_1)] \qquad \text{(Eq 6)}$$

To discuss the range of validity of Eq 4 and 5, the dependence of tangent Young's modulus $E_{TAN} = \mathrm{d}\sigma/\mathrm{d}e$ on stress will be used,

$$E_{TAN} = E_1/[1 + 2\mathrm{C}\ (\sigma/E_1)] \qquad \text{(Eq 7)}$$

The value of E_{TAN} increases with increasing compressive stress, $\sigma < 0$, and for finite positive values of E_{TAN} the relation $\sigma/E_1 > -1/(2\mathrm{C})$ must be satisfied. A limit value σ_L of the compressive stresses will be chosen as $\sigma_L/E_1 = -1/(\mathrm{nC})$ with $\mathrm{n} > 2$. The limit stress corresponds to the case where nearly all the microcracks have been closed and the tangent modulus from Eq 7 reaches its maximum value $E_{LTAN} = [\mathrm{n}/(\mathrm{n} - 2)]\ E_1$.

Preliminary estimates of the limit tangent modulus based on the measurement of ultrasound velocity in dependence on hydrostatic pressure suggest the relation $E_{LTAN} = 3E_1$ for plasma sprayed alumina (the increase from $E_1 \cong 60$ GPa to $E_{LTAN} \cong 180$ GPa), i.e., n = 3. A similar effect of uniaxial compression can be expected. The limit value of compressive stresses is then $(\sigma_L/E_1) = -1/(3\mathrm{C})$ and the limit value of elastic deformation follows from Eq 4 as $e_L = -2/(9\mathrm{C})$.

The value C = 30 seems reasonable for ceramic materials.[10] The limit value of stress is then $(\sigma_L/E_1) = -1/90 = -0.0111$ (e.g., for $E_1 = 60$ GPa, $\sigma_L = -670$ MPa) and the limit value of elastic strain $e_L = -2/270 = -7.41 \times 10^{-3}$.

The stress-strain relation given by Eq 4 or 5, proposed for a typical plasma sprayed ceramics like alumina with C = 30 and n = 3, is shown in Fig. 2. The corresponding dependences of tangent and secant Young's moduli on stress are given in Fig. 3. If necessary, the stress-strain relation can be extrapolated to higher compressive stresses with constant tangent Young's modulus, $(\sigma - \sigma_L) = E_{LTAN}\ (e - e_L)$, as shown by the dashed line on the left-hand side in Fig. 2 and 3. The stress-strain relation can also be extrapolated as an approximation to small tensile stresses $0 < \sigma < \sigma_F$ [where σ_F is the fracture stress, for plasma sprayed ce-

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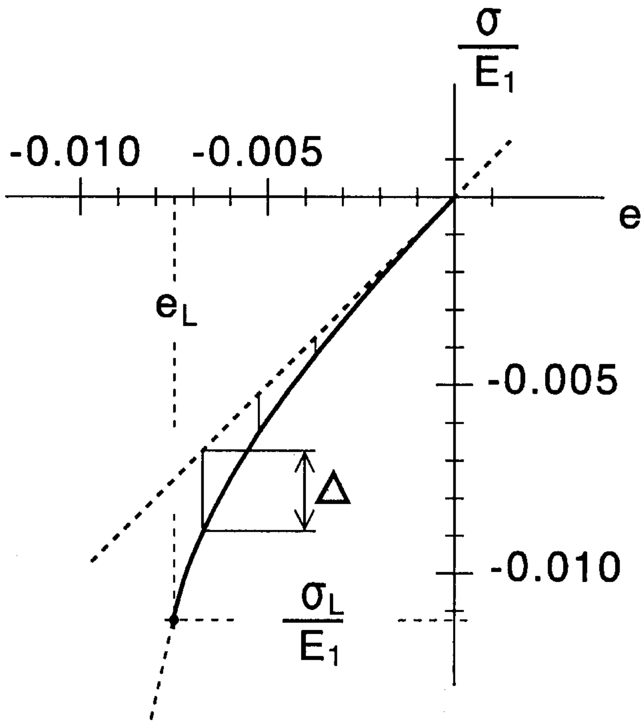


Fig. 2 Stress-strain relation (Eq 5) for C = 30 and n = 3; Δ shows the difference between the non-linear and linear laws

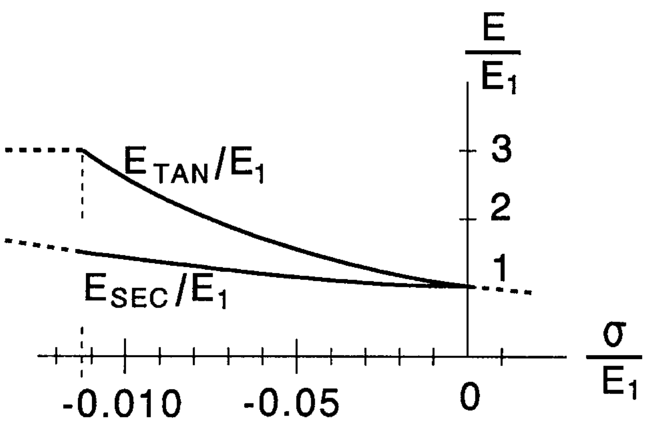


Fig. 3 Dependence of E_{TAN} and E_{SEC} on stress σ for C = 30 and n = 3

ramic materials rather low, $(\sigma_F/E_1) \cong 10^{-3}$] as shown by the dashed line on the right-hand side in Fig. 2 and 3.

The unknown constants A and B can be calculated from the conditions of equilibrium of forces and moments over the cross section x = constant,

$$\int_{-h_S}^{0} \sigma_S(z)\mathrm{d}z + \int_{0}^{h} \sigma(z)\mathrm{d}z = 0 \qquad \text{(Eq 8)}$$

$$\int_{-h_S}^{0} \sigma_S(z)z\mathrm{d}z + \int_{0}^{h} \sigma(z)z\mathrm{d}z = M \qquad \text{(Eq 9)}$$

Eq 8 and 9 (with Eq 3, 5, and 2) form a set of two non-linear equations for the two unknown constants A and B.

Solving for A and B, the stresses $\sigma_S(z)$ and $\sigma(z)$ follow from Eq 3 and 5 with Eq 2. The radius of curvature R of the beam can be computed as

$$(R/h) = (1/\mathrm{A}) \qquad \text{(Eq 10)}$$

2.2 Analytical Solution for Linear Coating (C = 0)

An analytical solution for C = 0 was established by several authors, e.g., Ref. 11-13. The integrals in Eq 8 and 9 are elementary, and the equations transform into a system of two linear algebraic equations for $\mathrm{A} = \mathrm{A}_0$ and $\mathrm{B} = \mathrm{B}_0$. Using dimensionless quantities

$$K = E_S/E_1,\ k = h_S/h,\ m = M/(h^2E_1) \qquad \text{(Eq 11)}$$

the results can be written separately for two particular cases.

1) For the effect of external moment $M \neq 0$ (and $\varepsilon = 0$):

$$\mathrm{A}_0 = 12[(1+Kk)/d]m \qquad \text{(Eq 12)}$$
$$\mathrm{B}_0 = 6[(Kk^2 - 1)/d]m$$

where the denominator $d > 0$,

$$d = 1 + K^2k^4 + 2Kk(2 + 3k + 2k^2) \qquad \text{(Eq 13)}$$

2) For the effect of residual stresses $\varepsilon \neq 0$ (and $M = 0$):

$$\mathrm{A}_0 = 6[Kk(1 + k)/d]\ \varepsilon, \qquad \text{(Eq 14)}$$
$$\mathrm{B}_0 = \{[1 + Kk^2(3 + 4k)]/d\}\ \varepsilon$$

In case C = 0, all the stresses in the substrate and in the coating are linear functions of z,

$$\sigma_S = E_S e_S,\ \sigma = E_1 e \qquad \text{(Eq 15)}$$

where e_S and e are given by Eq 2 with A_0, B_0 written instead of A, B.

2.3 Non-linear Coating (C ≠ 0)

With C≠0, the integrals in Eq 8 and 9 can again be evaluated analytically and the equations transform into a system of two non-linear equations for the constants A and B,

$$K[-(k^2/2)\mathrm{A} + k\mathrm{B}] - 1/(2\mathrm{C}) + [1/(12\mathrm{C}^2\mathrm{A})]$$
$$\{[1 + 4\mathrm{C}(\mathrm{A} + \mathrm{B} - \varepsilon)]^{3/2} - [1 + 4\mathrm{C}(\mathrm{B} - \varepsilon)]^{3/2}\} = 0 \qquad \text{(Eq 16)}$$

$$K[(k^3/3)\mathrm{A} - (k^2/2)\mathrm{B}] - 1/(4\mathrm{C}) + [1/(16\mathrm{C}^3\mathrm{A}^2)]$$
$$\{(1/5)[1+4\mathrm{C}(\mathrm{A}+\mathrm{B} - \varepsilon)]^{5/2} + (2/15)[1 + 4\mathrm{C}(\mathrm{B} - \varepsilon)]^{5/2}$$
$$- (1/3)[1 + 4\mathrm{C}(\mathrm{B} - \varepsilon)]\ [1 + 4\mathrm{C}(\mathrm{A} + \mathrm{B} - \varepsilon)]^{3/2}\} = m \qquad \text{(Eq 17)}$$

where the dimensionless quantities $K > 0$, $k > 0$ and m from Eq 11 are used. Note that external moment M leading to compressive stresses in the coating is negative (Fig. 1), i.e., $m < 0$. Initial deformation leading to compressive stresses in the coating is positive, $\varepsilon > 0$. The corresponding radii of curvature $R_M < 0$ and $R_\varepsilon > 0$ are shown in Fig. 1.

Table 1 Values of Constants A, B (Solution of Non-linear Eq 16 and 17) and A_0, B_0 (Linear Eq 12 and 14), Due to Dimensionless Moments $m = M/(h^2E_1) < 0$, $\varepsilon = 0$, or Due to Initial Strains $\varepsilon > 0$, $m = 0$ for $C = 30$, $K = E_S/E_1 = 2$ and for Two Values of k

	$k = h_S/h = 4$				$k = h_S/h = 2$			
$10^2\ m$	−0.60530	−1.86100	−3.20300	−4.6970	−0.19210	−0.5962	−1.0370	−1.5410
$10^2\ A_0$	−0.03712	−0.11420	−0.19640	−0.2881	−0.05971	−0.1853	−0.3224	−0.4790
$10^2\ B_0$	−0.06394	−0.19660	−0.33830	−0.4961	−0.04179	−0.1297	−0.2257	−0.3353
10^2 A	−0.03682	−0.11110	−0.18650	−0.2640	−0.05902	−0.1784	−0.3003	−0.4263
10^2 B	−0.06318	−0.18890	−0.31350	−0.4360	−0.04098	−0.1216	−0.1997	−0.2737
$10^2\ \varepsilon$	0.1	0.3	0.5	0.7	0.1	0.3	0.5	0.7
$10^2\ A_0$	0.01377	0.04089	0.06814	0.0954	0.03731	0.1119	0.1865	0.2611
$10^2\ B_0$	0.03458	0.10370	0.17290	0.2421	0.04611	0.1383	0.2306	0.3228
10^2 A	0.01377	0.04222	0.07202	0.1034	0.03744	0.1131	0.1900	0.2681
10^2 B	0.03494	0.10710	0.18280	0.2625	0.04629	0.1400	0.2352	0.3321

The two non-linear equations were solved numerically[1] for C = 20, 30, 40 and different values of K, k, m (with $\varepsilon = 0$) and ε (with $m = 0$). As an example, the results for C = 30, $K = E_S/E_1 = 2$, for two values of $k = h_S/h$, $k = 4$ and $k = 2$, and for increasing values of m and ε will be presented and compared with the linear solution.

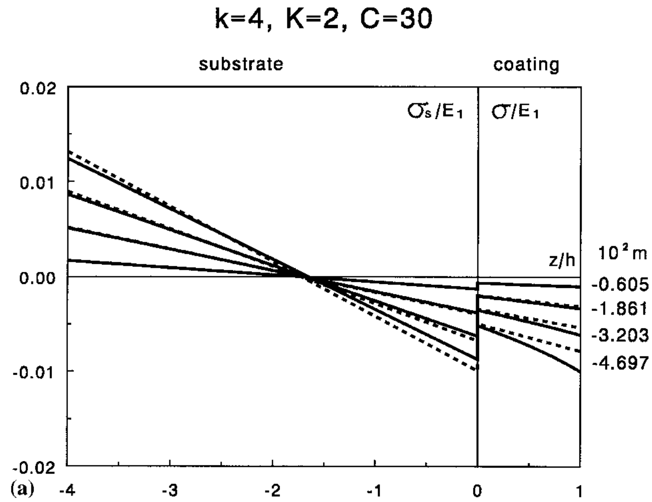
The computed values of constants A, B, together with the values of A_0, B_0 from Eq 12-14, are given in Table 1. The dependence of normalized stresses σ/E_1 and σ_S/E_1 on normalized coordinate z/h are shown in Fig. 4 for $m \neq 0$ and in Fig. 5 for $\varepsilon \neq 0$. The stresses corresponding to the non-linear solution (given by Eq 3 and 5 with Eq 2) are plotted by full lines and the stresses for the linear solution (given by Eq 15 with Eq 2 for A_0, B_0) are plotted by dashed lines.

3. Discussion

The non-linear elastic stress-strain relation given by Eq 4 or 5 is based on a simple model of the closing of microcracks by compressive stresses in thermally sprayed materials from Ref 10. This non-linearity is best manifested by the steep increase of tangent Young's modulus E_{TAN} with increasing compressive stresses (Eq 7 and Fig. 3), confirmed directly by measurements of ultrasound velocity where very small stress increments are superimposed on the compressive pre-stress.

However, the increase of compressive stresses with deformation is less steep and the non-linear solution gives only a modest correction to the linear solution, as shown for bending of substrates with sprayed coatings in Fig. 4 and 5.

Such correction can well be illustrated by the analysis of residual stresses in a thin coating on a thick substrate, for $k = h_S/h >> 1$, for initial deformation $\varepsilon > 0$ and for $m = 0$. This case can be treated as a coating on a half-space (a thick substrate with neglected bending). The homogeneous elastic deformation e of the coating is then given, $e = -\varepsilon$ (see e.g., Ref. 13). The dependence of the homogeneous compressive stress σ on e from Eq 5 is shown in Fig. 2 for the model material with C = 30. The difference Δ between the non-linear and linear stresses changes with increasing deformation, from a few percent to dozens of percent. The stress in the non-linear solution is proportional to the current value of the secant modulus, $\sigma = E_{SEC}(\sigma)e$, and the



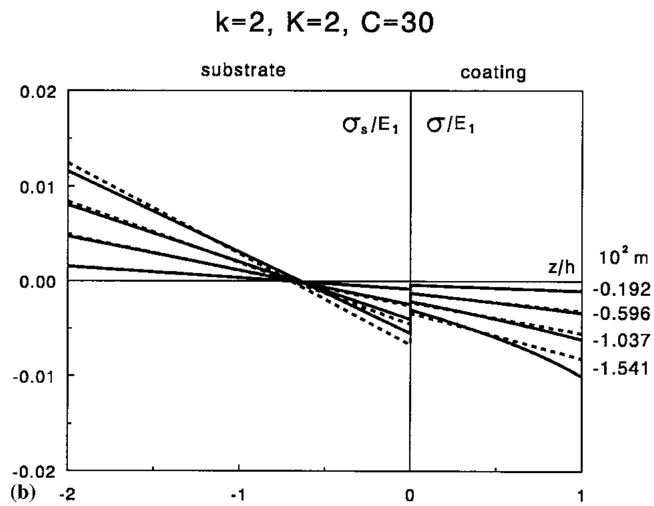
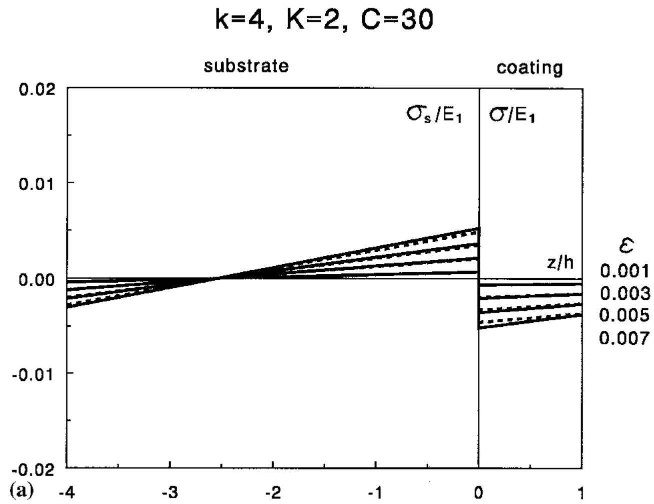


Fig. 4 Dependence of normalized stresses σ/E_1 and σ_s/E_1 on normalized coordinate z/h for four values of dimensionless external moment $m = M/(h^2E_1) < 0$ (without initial strain, $\varepsilon = 0$); full lines, non-linear solution; dashed lines, linear solution. Dimensionless parameters $K = E_s/E_1 = 2$ and $k = h_s/h = 4$ or 2 are chosen.

stress dependence $E_{SEC}(\sigma)$ is less steep than that of $E_{TAN}(\sigma)$ (Fig. 3). Even for the limit deformation $e_L = -0.0074$, i.e., for the limit stress $\sigma_L/E_1 = -0.0111$, where the current values of

[1]Bisection algorithm was used. A Fortran program is printed out in the Appendix.



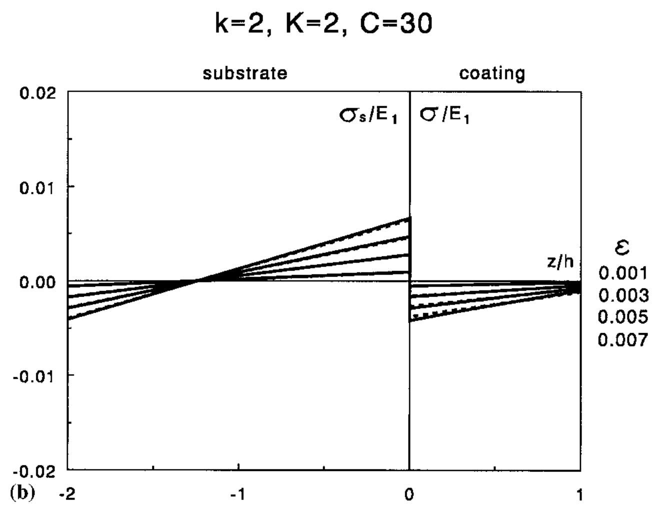


Fig. 5 Residual stresses for four values of thermal strain in the coating, $\varepsilon > 0$ (without external moment, $m = 0$); full lines, non-linear solution; dashed lines, linear solution

Young's moduli are $E_{TAN}/E_1 = 3$ but $E_{SEC}/E_1 = 1.5$ only, the non-linear stress is only 50% larger than the stress following from linear Hooke's law for the same deformation, $\sigma/E_1 = e_L$.

The position z_0 of the neutral axis follows from Eq 1 with the condition $e^T(z_0) = 0$. It is interesting to note that within linear theory $z_0/h = -B_0/A_0$ does not depend, according to Eq 11-14, on external moment m or on initial deformation ε and depends only on ratios K and k. On the other hand, the position of the neutral axis in the non-linear case, $z_0/h = -B/A$, depends slightly on m or ε.

The deformation e in the stress-strain relation given by Eq 4 or 5 has been assumed to be elastic. However, even at the unidirectional compressive stress, small inelastic contribution caused by sliding along the microcrack faces by the shear stress components can be expected, as mentioned in Ref. 8 and 14.

The mechanical properties of plasma sprayed materials have usually been considered in literature as linear elastic (e.g., in Ref. 1). However, in general, the specific microstructure of these materials, especially the high density of microcracks and imperfect bonds between the splats, lead to non-linear elastic and inelastic effects, even at small deformations. They can be observed in coatings also under tensile deformations $0 < e < 0.1\%$,[15] before macrocracks are formed by interconnection of the microcracks. In this case, the non-linearity is caused by increasing microcrack densities, only partly by elastic processes (interfaces in close contact without bonding act as microcracks at tensile deformation) and mainly by inelastic effects (some weak bonds can be broken even by small tensile stresses so that some microcracks can grow, and new microcracks can be formed). Such tensile behavior could also be formally described by the stress-strain relation introduced in Eq 4.

On the other hand, the elastic deformation due to closing of the microcracks seems to be the main contribution to the non-linearity of sprayed materials under compressive stresses. It follows from the presented study of bending that the non-linear elastic behavior should be taken into account at high compressive stresses, especially in plasma-sprayed ceramic coatings and free-standing parts with high densities of microcracks.

4. Conclusions

- Young's moduli E_1 of thermally sprayed materials, especially of ceramics, subjected to small stresses are much smaller than Young's modulus E_0 of a well-sintered material. This effect is well known to be mainly due to the presence of a high density of microcracks.[1]
- The increase of Young's modulus E_1 to the values closer to E_0 under higher compressive stresses is due to the elastic closing of microcracks with small crack aspect ratios. This effect was predicted on the basis of theoretical considerations[6] and modeled within non-linear elasticity theory.[10]
- In this paper, the stress distribution in a beam composed of elastically non-linear coating and elastically linear substrate due to bending by external moments or residual stresses was calculated. The typical properties of a model ceramic coating were chosen. Corrections to previously published linear solutions become significant for high compressive stresses in the coating.
- The results can be applied to the improvement of evaluation of bending experiments, often used for the determination of elastic constants or residual stresses in the coatings.
- The microcrack densities and distributions of crack aspect ratios sensitively depend not only on the coating material but also on the parameters of the spraying technology. More quantitative microscopic experiments on microcracks as well as macroscopic measurements of the dependence of Young's moduli on compressive stress in thermally sprayed materials would be useful.

Acknowledgments

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APPENDIX

A Fortran program for the solution of Eq 16 and 7, using the bisection method.

```
PROGRAM LAYER
IMPLICIT INTEGER*4 (I-N), REAL*8 (A-H,O-Z)
************************************************************************
* *
* Input file: layer.i (example) *
* *
* Es/Eo CEo hs/h ! Problem data (compulsory comment line) *
* 2 30 4 ! real real real *
* eps eh flag ! Loading (compulsory comment line) *
* 5.e-3 0 1 ! real real integer *
* *
* Es Young's modulus of the beam *
* Eo Young's modulus of the coating *
* CEo e = (sigma + C*sigma^2)/Eo, CEo = C*Eo *
* hs height of the beam *
* h thickness of the coating *
* eps initial (thermal) strain in the coating *
* eh total strain on the top surface (z=h) *
* flag = 0 eps, eh prescribed, moment computed *
* = 1 eps, m=0 prescribed, eh ignored, deflection computed *
* *
* Output file: layer.o *
* *
************************************************************************
COMMON E,CE,H,EPS,EH
C
MR = 50
OPEN(MR,FILE = 'layer.i',STATUS='OLD')
REWIND MR
MW = 60
OPEN(MW,FILE = 'layer.o')
CLOSE(MW,STATUS='DELETE')
OPEN(MW,FILE = 'layer.o')
REWIND MW
C
C Read E=Es/Eo, CE=CEo, H=hs/h
READ(MR,'()')
READ(MR,*) E, CE, H
C Read EPS=eps, EH=eh, KEY=flag
READ(MR,'()')
READ(MR,*) EPS, EH, KEY
```

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C
C Write input
WRITE(MW,‘(12x,“*** Input data ***”)’)
WRITE(MW,‘(/7HEs/Eo =,F4.1,9H, CEo =,F5.1,10H, hs/h =,F4.1)’)
* E,CE,H
WRITE(MW,‘(/5Heps =,E12.4,8H, eh =,E12.4)’) EPS, EH
IF(KEY.EQ0) WRITE(MW,‘(/“Solve for eps, eh given”)’)
IF(KEY.EQ1) WRITE(MW,‘(/“Solve for zero moment (eh ignored)”)’)
C
IF(KEY.EQ0) CALL SOLVEH(A,B)
IF(KEY.EQ1) CALL SOLVEM(A,B)
RN = FORCE(B)
RM = COUPLE(B)
C
D = 1 + E*E*H**4 + 2*E*H*(2+3*H+2*H*H)
IF(KEY.EQ0) THEN
A0 = 12*(1+E*H)*RM/D
B0 = 6*(E*H*H-1)*RM/D
ELSEIF(KEY.EQ1) THEN
A0 = 6*E*H*(1+H)*EPS/D
B0 = (1+E*H*H*(3+4*H))*EPS/D
ENDIF
C
C Write results
WRITE(MW,‘(/12x,“*** Results ***”)’)
WRITE(MW,‘(/“total strain = A*(z/h) + B”)’)
WRITE(MW,‘(“A, B constants”)’)
WRITE(MW,‘(“z vertical coordinate (z=0 at the interface)”)’)
WRITE(MW,‘(“h thickness of the coating”)’)
WRITE(MW,‘(“N total axial force = 0”)’)
WRITE(MW,‘(“m bending moment = M/(width*Eo*h^2)”)’)
WRITE(MW,‘(/4HA =,E12.4)’) A
WRITE(MW,‘(4HB =,E12.4)’) B
WRITE(MW,‘(4HA0 =,E12.4)’) A0
WRITE(MW,‘(4HB0 =,E12.4)’) B0
WRITE(MW,‘(4HN =,E12.4)’) RN
WRITE(MW,‘(4Hm =,E12.4)’) RM
WRITE(MW,‘(/12X,“*** End of layer ***”)’)
STOP ‘O.K.’
END
SUBROUTINE SOLVEH(A,B)
***********************************************************************
* *
* Solve for constants A, B if EH is given. *
* *
* Input: *
* Problem data, EH ... common entries *
* *
* Output: *
* A, B *
* *
***********************************************************************
IMPLICIT INTEGER*4 (I-N), REAL*8 (A-H,O-Z)
COMMON E,CE,H,EPS,EH
EXTERNAL FORCE
CRIT = 1/(4*CE)*0.9
BTOL = 1.E-14*CRIT
B1 = EPS - CRIT
B2 = EPS + CRIT
ROOT = FORCE(B1)*FORCE(B2)
```

```
IF(EH-EPS.GT.CRIT .OR. ROOT.GE.0) STOP 'Out of range'
CALL DIVINT(B1,B2,BTOL,FORCE, B)
A = EH - B
END
SUBROUTINE SOLVEM(A,B)
**************************************************************************
* *
* Solve for constants A, B if m = 0. *
* *
* Input: *
* Problem data ... common entries *
* *
* Output: *
* A, B *
* *
**************************************************************************
IMPLICIT INTEGER*4 (I-N), REAL*8 (A-H,O-Z)
COMMON E,CE,H,EPS,EH
EXTERNAL EQUI
EHTOL = 1.E-14*ABS(EPS)
IF(EPS.GT.0) THEN
EH1 = 0
EH2 = EPS
ELSEIF(EPS.LT.0) THEN
EH1 = EPS
EH2 = 0
ENDIF
CALL DIVINT(EH1,EH2,EHTOL,EQUI, EHH)
EH = EHH
CALL SOLVEH(A,B)
A = EH - B
END
FUNCTION EQUI(STRAIN)
**************************************************************************
* *
* Compute moment if STRAIN is given. *
* *
**************************************************************************
IMPLICIT INTEGER*4 (I-N), REAL*8 (A-H,O-Z)
COMMON E,CE,H,EPS,EH
EH = STRAIN
CALL SOLVEH(A,B)
EQUI = COUPLE(B)
END
FUNCTION FORCE(B)
**************************************************************************
* *
* Compute axial force. *
* *
**************************************************************************
IMPLICIT INTEGER*4 (I-N), REAL*8 (A-H,O-Z)
COMMON E,CE,H,EPS,EH
A = EH - B
C1 = 1 + 4*CE*(A+B-EPS)
C2 = 1 + 4*CE*(B-EPS)
FORCE = (C1**1.5 - C2**1.5)/(12*A*CE**2)
FORCE = E*(B*H-A*H**2/2) - 1/(2*CE) + FORCE
END
FUNCTION COUPLE (B)
**************************************************************************
```

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```
Compute bending moment.
**********************************************************************
IMPLICIT INTEGER*4 (I-N), REAL*8 (A-H, O-Z)
COMMON E, CE, H, EPS, EH
A = EH - B
C1 = 1 + 4*CE*(A+B-EPS)
C2 = 1 + 4*CE*(B-EPS)
COUPLE = (C1**2.5/5 + 2*C2**2.5/15 - C2*C1**1.5/3)/(16*A**2*CE**3)
COUPLE = E* (A*H**3/3-B*H**2/2)-1/(4*CE) + COUPLE
END
SUBROUTINE DIVINT (X1, X2, TOL, F, X0)
**********************************************************************
Interval division.
Input:
X1, X2 interval
TOL tolerance prescribed
F (X) external function
Output:
X0 root
**********************************************************************
IMPLICIT INTEGER*4 (I-N), REAL*8 (A-H, O-Z)
F1 = F (X1)
1 X0 = (X1+X2)/2
IF (X2-X1.LT.TOL) RETURN
F0 = F (X0)
IF (F1*F0.LT.0) THEN
X2 = X0
ELSE
X1 = X0
F1 = F0
ENDIF
GOTO 1
END
```